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The $O(\alpha\alpha_s)$ correction to the pole mass of the t-quark within the Standard Model

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Abstract

We have calculated the $O(\alpha\alpha_s)$ contributions to the relationship between the $\overline{\mathrm{MS}}$–mass and the pole of the t-quark propagator in the Standard Model in the limit of a diagonal CKM matrix and for a massless b-quark. Analytical results for the so far unknown master–integrals appearing in the calculation are also given.

1 Introduction

Electroweak precision observables play an important role in the verification of the Standard Model (SM) at the quantum level. By comparing precision data with corresponding predictions it is possible to get constraints on unknown parameters, like the Higgs boson mass, or unveil new physics. For electroweak processes perturbative calculations work and converge rather well. However, as the complexity of such calculations grows dramatically with the order of the perturbation expansion, complete higher order results are available in a few cases only. Therefore, perturbative calculations are possible only at limited accuracy. We usually distinguish two types of uncertainties: unknown higher order effects and the experimental[1] errors of the input parameters. For the calculation of electroweak observables the generally accepted renormalization scheme, defining a particular parametrization, is the so called *on-shell scheme* [1]– [5], where, in addition to the fine structure constant (and/or the Fermi constant), the pole masses of particles serve as input parameters. Quark masses require special consideration in this context, since on–shell quark masses are not accessible

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[1]In general they also include systematic errors which stem from non–negligible theoretical uncertainties.

experimentally. Fortunately, the light quark masses in high energy processes often can be neglected (effects $O(m_q^2/M_Z^2)$) and thus can be treated as massless in practical calculations. The top quark is different. The large numerical value of the top quark mass in conjunction with the violation of the Appelquist-Carazzone theorem [6] as a consequence of the Higgs mechanism of mass generation, implies that a class of radiative corrections are proportional to positive powers of the top-quark mass which gives rise to sizeable effects. Moreover, the concept of a pole mass of a quark is intrinsically ambiguous due to strong interaction renormalon contributions [7].

The present experimental error of the top-quark mass, ~ 5 GeV, will be reduced to 1-2 GeV at the LHC and/or at a future Linear Collider. The question about which type of mass will be determined actually with this accuracy is discussed in [8]. The higher order theoretical uncertainties may be estimated from scheme– and scale–variations [9].

The aim of the present paper is the analytical calculation of the $O(\alpha\alpha_s)$ correction to the relationship between the pole– and the $\overline{\rm MS}$–mass of the top–quark. This also provides us the two–loop on–shell mass counter-term for the top quark. The one–loop results of order $O(\alpha_s)$ and $O(\alpha)$ have been presented in [10] and e.g. in [4] [2], respectively. The one-loop $O(\alpha)$ correction to the relationship between the top-Yukawa coupling and the pole mass of the top-quark has been calculated in [11]. The two–loop $O(\alpha_s^2)$ correction is given in [12], and the same order result obtained via regularization by dimensional reduction may be found in [13]. The renormalized off-shell fermion propagator of order $O(\alpha_s^2)$ has been worked out in [14]. Only recently, in [15], the three–loop $O(\alpha_s^3)$ correction has been published. Finally, the two–loop $O(\alpha\alpha_s)$ and $O(\alpha^2)$ corrections have been calculated in the approximation of vanishing electroweak gauge couplings [16]. Our calculation, presented here, extends previous two–loop $O(\alpha\alpha_s)$ calculations of the gauge boson self-energies [17] and the SM $O(\alpha^2)$ corrections to the relation between the pole and the $\overline{\rm MS}$ masses of the gauge bosons Z and W, presented in [18, 19].

The paper is organized as follows: in Sec. 2 we outline the calculation of the on–shell fermion self–energy. The relevant master–integrals are presented in Sec. 3. In Sec. 4 we discuss the renormalization problems at two–loops. Sec. 5 contains the main results of our calculation and in Sec. 6 we summarize our conclusions.

2 The quark pole mass: definition and calculation

The definition of the top-quark pole mass has been discussed in [20]. Starting point of our consideration is the tensor decomposition of the one–particle irreducible self–energy of a massive fermion $\tilde{\Sigma}(p, m, \ldots)$ which, within the SM, has the form

$$\tilde{\Sigma}(p,m,\ldots) \;=\; i\hat{p}\left[\tilde{A}(p^2,m,\ldots)-\gamma_5\tilde{C}(p^2,m,\ldots)\right]+m\left[\tilde{B}(p^2,m,\ldots)-\gamma_5\tilde{D}(p^2,m,\ldots)\right] \quad (2.1)$$

where $\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}$ are Lorentz scalar functions depending on all parameters of the SM. For simplicity of the notation we will indicate explicitly only the external momentum p^2 and the

[2](see also Eq. (B.5) in Appendix B of [19])

mass m of the quark under consideration. The $O(\alpha\alpha_s)$ correction has a structure similar to the one in QCD, where $\tilde{C} = \tilde{D} = 0$. In this case, the position of the pole $-\tilde{M}$ is defined as the formal solution for $i\hat{p}$ (in Euclidean metric [21] with $\hat{p} = \gamma p$, $\hat{p}^2 = p^2$; on–shell: $\hat{p} = im$, $p^2 = -m^2$) at which the inverse of the connected full propagator equals zero. Thus

$$i\hat{p} + m - \tilde{\Sigma}(p, m, \ldots) = i\hat{p}\ \left(1 - \tilde{A}(p^2, m, \ldots)\right) + m\ \left(1 - \tilde{B}(p^2, m, \ldots)\right) = 0 \tag{2.2}$$

for $\hat{p} = i\tilde{M}$. The immediate question arising here is, what is the interpretation of the complex mass $\tilde{M} \equiv M' - \frac{i}{2}\Gamma'$? In general in a perturbative calculation a transition amplitude T exhibiting an unstable particle resonance has the form (U, V some spinor valued amplitudes)

$$T = \bar{U}\ \frac{1}{i\hat{p} + \tilde{M}\ (1 + O(p^2 + \tilde{M}^2))}\ V = \frac{\bar{U}\ (-i\hat{p} + \tilde{M})\ V}{p^2 + \tilde{M}^2 + O((p^2 + \tilde{M}^2)^2)} = \frac{R}{p^2 + \tilde{M}^2} + B(p^2) \tag{2.3}$$

where R is a complex number and B is a background complex scalar amplitude which is regular at $p^2 = -\tilde{M}^2$. T thus has the same general form as in the case of a bosonic resonance. We thus define the pole mass M and the on–shell width Γ as in the bosonic case by

$$\tilde{M}^2 = M^2 - iM\Gamma = M^{'2} - \Gamma^{'2}/4 - iM'\Gamma' \tag{2.4}$$

such that

$$M = \sqrt{M'^2 - \Gamma'^2/4}\ \ ; \quad \Gamma = \frac{M'}{M}\Gamma' \tag{2.5}$$

Since $M = M' + O(\alpha^2)$ and $\Gamma = \Gamma' + O(\alpha^2)$ for the $O(\alpha\alpha_s)$ terms considered in this paper we can identify $M = M'$ and $\Gamma = \Gamma'$ in the following.

For the remainder of the paper we will adopt the following notation: capital $M \simeq \mathrm{Re}\,\tilde{M}$ always denotes the pole mass; lower case m stands for the renormalized mass in the $\overline{\mathrm{MS}}$ scheme, while m_0 denotes the bare mass. The on–shell width is given by $\Gamma \simeq -2\,\mathrm{Im}\,\tilde{M}$. In addition we use e, g and g_s to denote the $U(1)_{\mathrm{em}}$, $SU(2)_{\mathrm{L}}$ and $SU(3)_{\mathrm{c}}$ couplings of the SM in the $\overline{\mathrm{MS}}$ scheme.

In perturbation theory (2.2) is to be solved order by order. For this aim we expand the self–energy function about the lowest order solution $\hat{p} = im_0$:

$$\tilde{\Sigma}(p, m, \ldots) \ \ = \ \ \tilde{\Sigma}\ |_{\hat{p}=im_0} + (i\hat{p} + m_0)\ \left[\tilde{\Sigma}'\right]\Big|_{\hat{p}=im_0} + \cdots \tag{2.6}$$

and define dimensionless "on–shell" amplitudes Σ, Σ' by

$$\tilde{\Sigma}\Big|_{\hat{p}=im_0} = \left[-m_0\tilde{A} + m_0\tilde{B}\right]\Big|_{p^2=-m_0^2} \equiv -m_0\Sigma(m_0, \ldots) \tag{2.7}$$

and

$$\left[\tilde{\Sigma}'\right]\Big|_{\hat{p}=im_0} = \left[\left(\frac{\partial\tilde{\Sigma}}{\partial(i\hat{p})}\right)\right]\Bigg|_{\hat{p}=im_0} = \left[\tilde{A} + 2p^2\dot{\tilde{A}} + 2m_0^2\dot{\tilde{B}}\right]\Big|_{p^2=-m_0^2} \equiv \Sigma'(m_0, \ldots) \tag{2.8}$$

where $\dot{X}(p^2,\ldots)$ denotes the derivative of $X(p^2,\ldots)$ with respect to p^2. To two loops we then have the solution (in agreement with [20])

$$\frac{\tilde{M}}{m} = 1 + \Sigma_1 + \Sigma_2 + \Sigma_1 \Sigma_1^{'} \tag{2.9}$$

where Σ_L is the bare ($m = m_0$) or $\overline{\mathrm{MS}}$–renormalized (m the $\overline{\mathrm{MS}}$–mass) L-loop contribution to the amplitudes defined in (2.7) and (2.8).

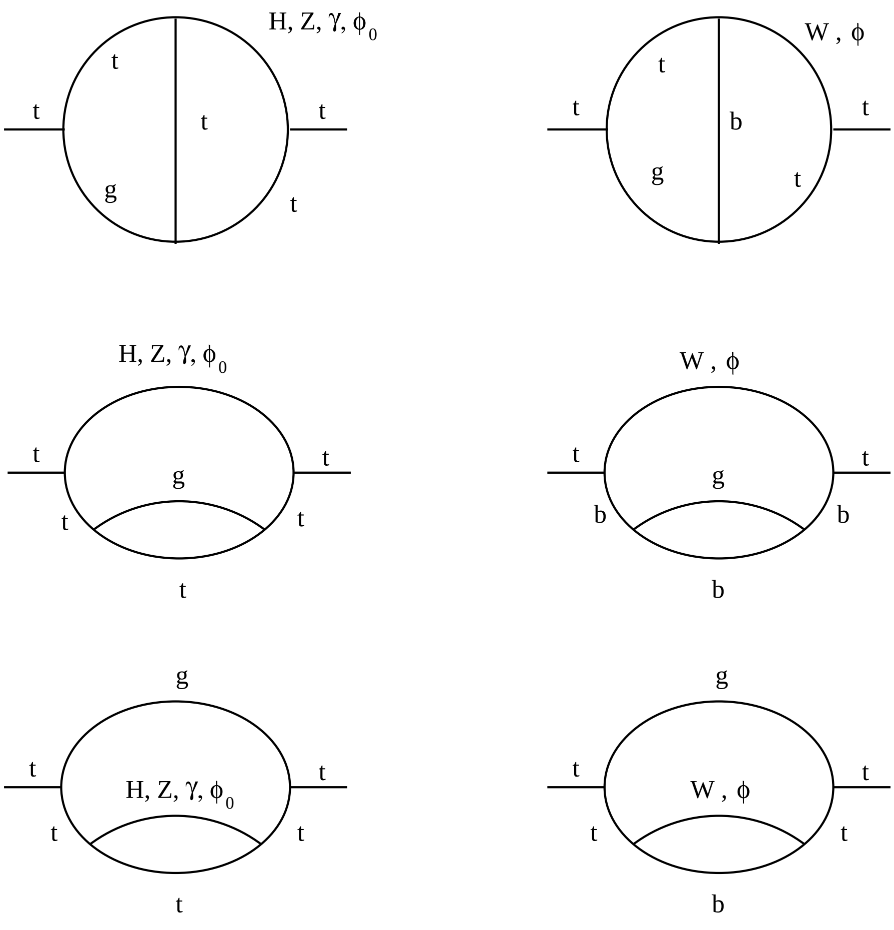

Figure 1: The two–loop one–particle irreducible diagrams contributing to the pole mass of a quark. ϕ_0 and ϕ are the neutral and the charged pseudo–Goldstone bosons, respectively. The number of diagrams is 24.

According to Eq. (2.9) we need to calculate propagator-type diagrams up to two loops on–shell. The set of two–loop one–particle irreducible diagrams is shown in Fig. 1. In order to get manifestly gauge invariant results the Higgs tadpole diagrams, shown in Fig. 2 on the left side, should be included [2]. As was demonstrated in [19] for the self–consistency of the renormalization group (RG) approach the two–loop tadpoles of the type shown on the right side of Fig. 2 should be included as well. For each quark species, there is one $O(\alpha\alpha_s)$ two-loop tadpole diagram which is gauge invariant. For a quark with mass m_q its

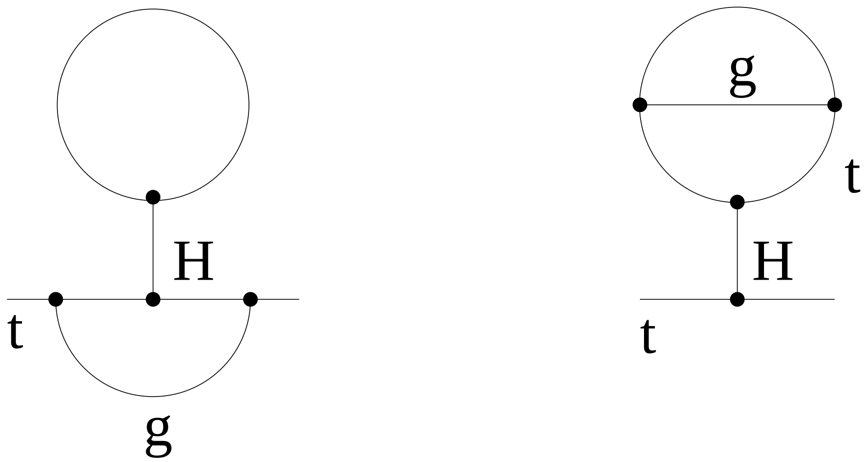

Figure 2: The two–loop tadpole type diagrams which should be included for manifest gauge and renormalization group invariance.

bare contribution to the location of the pole of the top–quark propagator reads (see Sec. 4.3 in [19])

$$\Delta_t = -\frac{g}{16\pi^2}\frac{g_s^2}{16\pi^2}16N_cC_f\frac{1}{(d-4)^2}\frac{(d-1)}{(d-3)}\frac{m_q^4}{m_W}(m_q^2)^{-2\varepsilon} ,$$

with $C_f = 4/3$ in the SM and N_c is the number of colors ($N_c = 3$). The one–loop $\mathcal{O}(\alpha)$ result for the pole mass of a quark in the SM, in the approximation of a diagonal Cabbibo-Kobayashi-Maskawa matrix is given by Eq. (B.5) in Appendix B of [19]. For the calculation of the two–loop propagator type diagrams with several mass scales we will use Tarasov's recurrence relations [22] which allow us to reduce all diagrams to a few master–integrals. The package **ONSHELL2** [23] is used for the calculation of the single scale diagrams.

3 Master–integrals

This section is devoted to the calculation of the so far unknown two-loop master-integrals needed for our calculation and shown in Fig. 3. We denote all master–integrals by $T_{AB\cdots}$, where the first letter $T = F, V, J$ indicates the topology in accordance with the notation introduced in [22]; indices $A, B, \cdots = 0, 1, 2$ characterize the relation between the corresponding internal mass to the external momentum: 0 indicates a massless line, 1 corresponds to "internal mass equal to external momentum" and 2 means that mass and momentum are different (see Fig. 3 for details). In our normalization each loop is divided by $(4\pi)^{2-\varepsilon}\Gamma(1+\varepsilon)$. We will also use the short notations

$$J_{m_1m_2m_3} = \frac{\pi^{-n}}{\Gamma^2\left(3-\frac{n}{2}\right)}\int \frac{d^nk_1d^nk_2}{[(k_1-p)^2+m_1^2][k_2^2+m_2^2][(k_1-k_2)^2+m_3^3]}\Bigg|_{p^2=-m^2} ,$$

$$A_0(M) \quad = \quad \frac{\pi^{-n/2}}{\Gamma\left(3-\frac{n}{2}\right)} \int \frac{d^n k_1}{k_1^2+M^2} \equiv \frac{4M^{n-2}}{(n-2)(n-4)} \,. \tag{3.10}$$

for the auxiliary integrals appearing in our calculation.

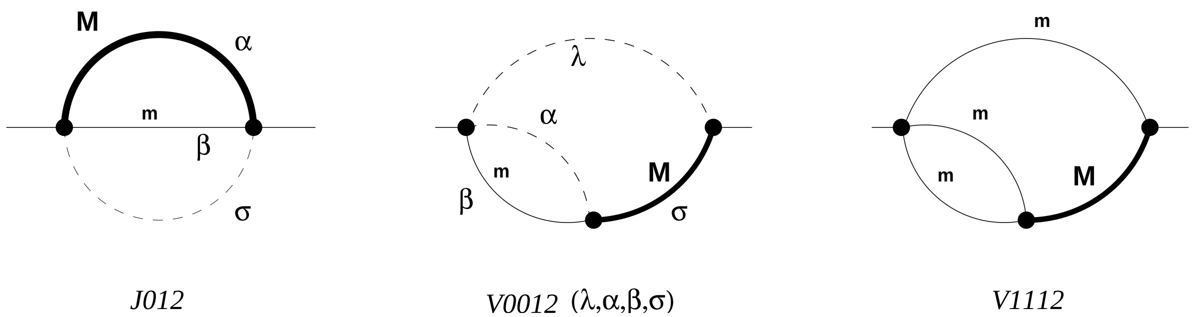

Figure 3: The master diagrams arising in this two–loop calculation. Bold, thin and dashed lines correspond to off–shell massive, on–shell massive and to massless propagators, respectively.

3.1 J_{012}

The analytical result for this type of integral for arbitrary values of the internal masses, momentum and powers of the propagators can be calculated by using the Mellin–Barnes technique [25]. It is expressible in terms of Appell hypergeometric function F_4 [26]:

$$\begin{aligned}
&J_{012}(\sigma,\beta,\alpha,p^2,m^2,M^2) = \int \frac{d^n k_1 d^n k_2}{((p-k_1)^2)^\sigma((k_1-k_2)^2+M^2)^\alpha(k_2^2+m^2)^\beta} \\
&= (M^2)^{n-\sigma-\alpha-\beta} \frac{\Gamma(\frac{n}{2}-\sigma)}{\Gamma(\sigma)\Gamma(\alpha)\Gamma(\beta)\Gamma(\frac{n}{2})\Gamma^2(3-\frac{n}{2})} \Bigg\{ \\
&\Gamma\left(\tfrac{n}{2}-\beta\right)\Gamma\left(\alpha+\beta+\sigma-n\right)\Gamma\left(\beta+\sigma-\tfrac{n}{2}\right) F_4\left(\sigma+\beta-\tfrac{n}{2}, \alpha+\beta+\sigma-n; \tfrac{n}{2}, 1+\beta-\tfrac{n}{2} \middle| \frac{-p^2}{M^2}, \frac{m^2}{M^2}\right) \\
&+ \left(\frac{m^2}{M^2}\right)^{(n/2-\beta)} \Gamma\left(\beta-\tfrac{n}{2}\right)\Gamma(\sigma)\Gamma\left(\alpha+\sigma-\tfrac{n}{2}\right) F_4\left(\sigma+\alpha-\tfrac{n}{2}, \sigma; \tfrac{n}{2}, 1+\tfrac{n}{2}-\beta \middle| \frac{-p^2}{M^2}, \frac{m^2}{M^2}\right)\Bigg\}
\end{aligned} \tag{3.11}$$

The result is symmetric with respect to the exchange $M^2 \leftrightarrow m^2$, $\alpha \leftrightarrow \beta$. The ε-expansion for this diagram up to the finite part has been given in [27] and was recently recalculated in [28]. In our case we need also the term linear in ε, however, only for the particular case, when the external momentum is on-shell with respect to one of the internal masses. In this case, taking $p^2 = -m^2$ (on–shell index β), we simplify our notation and write

$$J_{012}(\sigma,\beta,\alpha) = J_{012}(\sigma,\beta,\alpha,p^2,m^2,M^2)\Big|_{p^2=-m^2} \,.$$

We thus consider

$$J_{012}(\sigma,\beta,\alpha) = (M^2)^{n-\sigma-\alpha-\beta}\frac{\Gamma(\frac{n}{2}-\sigma)}{\Gamma(\sigma)\Gamma(\alpha)\Gamma(\beta)\Gamma(\frac{n}{2})\Gamma^2(3-\frac{n}{2})}\Bigg\{$$
$$\Gamma\left(\tfrac{n}{2}-\beta\right)\Gamma\left(\alpha+\beta+\sigma-n\right)\Gamma\left(\beta+\sigma-\tfrac{n}{2}\right){}_4F_3\left(\begin{array}{c}\alpha+\beta+\sigma-n,\beta+\sigma-\frac{n}{2},\frac{\beta}{2},\frac{1+\beta}{2}\\ 1+\beta-\frac{n}{2},\beta,\frac{n}{2}\end{array}\Bigg|\frac{4m^2}{M^2}\right)$$
$$+\left(\frac{m^2}{M^2}\right)^{(n/2-\beta)}\Gamma\left(\beta-\tfrac{n}{2}\right)\Gamma(\sigma)\Gamma\left(\alpha+\sigma-\tfrac{n}{2}\right){}_4F_3\left(\begin{array}{c}\sigma,\alpha+\sigma-\frac{n}{2},\frac{n-\beta}{2},\frac{1+n-\beta}{2}\\ 1+\frac{n}{2}-\beta,n-\beta,\frac{n}{2}\end{array}\Bigg|\frac{4m^2}{M^2}\right)\Bigg\} \quad (3.12)$$

in the following. Let us remind that for the class of integrals J_{012} there are three master–integrals of this type: $J_{012}(1,1,1)$, $J_{012}(1,2,1)$ and $J_{012}(1,1,2)$ [22]. However, other independent combinations happen to be more suitable for performing the ε-expansion: $J_{012}(1,2,2)$, $J_{012}(2,2,1)$, and $[J_{012}(1,2,2)+J_{012}(2,1,2)+J_{012}(2,2,1)]$ (see also in Ref. [29]). The latter combination corresponds to the integral $J_{012}(1,1,1)$ in $2-2\varepsilon$ dimensions [30] and we have:

$$\begin{aligned}J_{012}(1,2,2) &= -\frac{(M^2)^{-1-2\varepsilon}}{\varepsilon(1-\varepsilon)}\Bigg\{\frac{\Gamma(1-\varepsilon)\Gamma(1+2\varepsilon)}{\Gamma(1+\varepsilon)}{}_3F_2\left(\begin{array}{c}1,\frac{3}{2},1+2\varepsilon\\ 2,2-\varepsilon\end{array}\Bigg|\frac{4m^2}{M^2}\right)\\ &\quad -\left(\frac{m^2}{M^2}\right)^{-\varepsilon}{}_3F_2\left(\begin{array}{c}1,1+\varepsilon,\frac{3}{2}-\varepsilon\\ 2-\varepsilon,2-2\varepsilon\end{array}\Bigg|\frac{4m^2}{M^2}\right)\Bigg\}, \quad (3.13)\end{aligned}$$

$$\begin{aligned}J_{012}(2,2,1) = \frac{(M^2)^{-1-2\varepsilon}}{\varepsilon^2(1-\varepsilon)}&\Bigg\{\frac{(1+\varepsilon)\Gamma(1-\varepsilon)\Gamma(1+2\varepsilon)}{\Gamma(1+\varepsilon)}{}_4F_3\left(\begin{array}{c}1,\frac{3}{2},1+2\varepsilon,2+\varepsilon\\ 2,2-\varepsilon,1+\varepsilon\end{array}\Bigg|\frac{4m^2}{M^2}\right)\\ &-\left(\frac{m^2}{M^2}\right)^{-\varepsilon}{}_3F_2\left(\begin{array}{c}2,1+\varepsilon,\frac{3}{2}-\varepsilon\\ 2-\varepsilon,2-2\varepsilon\end{array}\Bigg|\frac{4m^2}{M^2}\right)\Bigg\}, \quad (3.14)\end{aligned}$$

$$\begin{aligned}J_c \equiv J_{012}(1,2,2)+J_{012}(2,1,2)+J_{012}(2,2,1) = (M^2)^{-1-2\varepsilon}\frac{1}{\varepsilon^2}\times\\ \Bigg\{\frac{\Gamma(1-\varepsilon)\Gamma(2+2\varepsilon)}{\Gamma(1+\varepsilon)}\left[1+2\frac{m^2}{M^2}\frac{1+2\varepsilon}{1-\varepsilon}{}_3F_2\left(\begin{array}{c}1,\frac{3}{2},2+2\varepsilon\\ 2,2-\varepsilon\end{array}\Bigg|\frac{4m^2}{M^2}\right)\right]\\ -\left(\frac{m^2}{M^2}\right)^{-\varepsilon}\left[1+2\frac{m^2}{M^2}\frac{1+\varepsilon}{1-\varepsilon}{}_3F_2\left(\begin{array}{c}1,2+\varepsilon,\frac{3}{2}-\varepsilon\\ 2-\varepsilon,2-2\varepsilon\end{array}\Bigg|\frac{4m^2}{M^2}\right)\right]\Bigg\}. \quad (3.15)\end{aligned}$$

The ε-expansion for the given hypergeometric functions were worked out in [19] and [31]. While the individual expressions are rather lengthy the results for the ε-expansion of the integrals have the compact form

$$\begin{aligned}J_{012}(1,2,2) &= (M^2)^{-1-2\varepsilon}\frac{(1+y)^2}{y}\Bigg\{\ln(1+y)\Big[\ln(1+y)-\ln(y)\Big]\\ &+\varepsilon\Big[2\ln^3(1+y)-2\zeta_2\ln(1+y)-3\ln y\ln^2(1+y)+\frac{1}{2}\ln^2 y\ln(1+y)\\ &-2\ln(y)\mathrm{Li}_2(-y)-3\ln(y)\mathrm{Li}_2(y)+6\mathrm{Li}_3(-y)+6\mathrm{Li}_3(y)\Big]+\mathcal{O}(\varepsilon^2)\Bigg\}, \quad (3.16)\end{aligned}$$

$$
\begin{aligned}
J_{012}(2,2,1) = {} & \frac{(M^2)^{-1-2\varepsilon}}{(1-\varepsilon)} \frac{(1+y)^2}{(1-y)} \left\{ \frac{1}{\varepsilon} \left[\frac{1-y}{y} \ln(1+y) + \ln y \right] - \frac{1}{2} \ln^2 y + 2\zeta_2 \right. \\
& +2\frac{1-y}{y} \ln(1+y) \Big(\ln(1+y) - \ln y \Big) - \frac{1+y}{y} \Big(3\ln y \ln(1-y) + 4\mathrm{Li}_2(-y) + 3\mathrm{Li}_2(y) \Big) \\
& +\varepsilon \left[\frac{1+y}{y} \Big(12\mathrm{S}_{1,2}\big(y^2\big) - 4\mathrm{S}_{1,2}(-y) - 6\mathrm{S}_{1,2}(y) + 24\ln(1-y)\mathrm{Li}_2(-y) \right. \\
& \quad +18\ln(1-y)\mathrm{Li}_2(y) + 9\ln y \ln^2(1-y) + \frac{3}{2}\ln^2 y \ln(1-y) - 6\zeta_2 \ln(1-y) \Big) \\
& \quad +\frac{1-y}{y} \Big(\frac{8}{3}\ln^3(1+y) - 2\zeta_2 \ln(1+y) - 4\ln y \ln^2(1+y) + \ln^2 y \ln(1+y) \Big) \\
& \quad +\frac{1}{6}\ln^3 y - 2\zeta_3 + \frac{15y+9}{y} \ln y \mathrm{Li}_2(y) + 4\frac{3y+2}{y} \ln y \mathrm{Li}_2(-y) \\
& \quad \left. \left. -4\frac{7y+4}{y} \mathrm{Li}_3(-y) - 3\frac{9y+5}{y} \mathrm{Li}_3(y) \right] + \mathcal{O}(\varepsilon^2) \right\} , \qquad (3.17)
\end{aligned}
$$

$$
\begin{aligned}
J_c \equiv {} & J_{012}(1,2,2) + J_{012}(2,1,2) + J_{012}(2,2,1) \\
= {} & (M^2)^{-1-2\varepsilon} \frac{1+y}{1-y} \left\{ \frac{1}{\varepsilon} \ln y - \left[6\ln y \ln(1-y) + \frac{1}{2}\ln^2 y - 2\zeta_2 + 8\mathrm{Li}_2(-y) + 6\mathrm{Li}_2(y) \right] \right. \\
& +\varepsilon \Big[24\mathrm{S}_{1,2}\big(y^2\big) - 8\mathrm{S}_{1,2}(-y) - 12\mathrm{S}_{1,2}(y) - 12\zeta_2 \ln(1-y) + 48\ln(1-y)\mathrm{Li}_2(-y) \\
& +36\ln(1-y)\mathrm{Li}_2(y) + 18\ln y \ln^2(1-y) + 3\ln^2 y \ln(1-y) + \frac{1}{6}\ln^3 y \\
& \left. +20\ln y \mathrm{Li}_2(-y) + 24\ln y \mathrm{Li}_2(y) - 2\zeta_3 - 44\mathrm{Li}_3(-y) - 42\mathrm{Li}_3(y) \Big] + \mathcal{O}(\varepsilon^2) \right\} , \qquad (3.18)
\end{aligned}
$$

where

$$
y = \frac{1-\sqrt{1-4\frac{m^2}{M^2}}}{1+\sqrt{1-4\frac{m^2}{M^2}}} , \qquad \frac{M^2}{m^2} = \frac{(1+y)^2}{y} . \qquad (3.19)
$$

The expressions for the original set of master-integrals then read:

$$
\begin{aligned}
J_{012}(1,1,1,m^2,M^2) = {} & M^2 \frac{[4m^2 - M^2(9n-32)]}{(3n-10)(3n-8)(n-3)} J_c \\
& + \frac{2(M^2-m^2)\left[4m^2(2n-7) + M^2(n-4)\right]}{(3n-10)(3n-8)(n-3)} \Big[J_{012}(2,2,1) + J_{012}(1,2,2) \Big] \\
& - A_0(m) A_0(M) \frac{(n-2)^2 \left[(7n-24)M^2 + m^2(8n-28)\right]}{4M^2 m^2 (3n-10)(3n-8)(n-3))} , \\
J_{012}(1,2,1,m^2,M^2) = {} & -\frac{2(M^2-m^2)(2n-7)}{(3n-10)(n-3)} J_{012}(1,2,2) - \frac{M^2}{(3n-10)(n-3)} J_c
\end{aligned}
$$

$$-\frac{[M^2(n-4)-2m^2(2n-7)]}{(3n-10)(n-3)}J_{012}(2,2,1)+A_0(m)A_0(M)\frac{(n-2)^2(2n-7)}{4M^2m^2(3n-10)(n-3)}\ ,$$

$$J_{012}(1,1,2,m^2,M^2)=-\frac{[2m^2(n-3)+M^2(n-4)]}{(3n-10)(n-3)}\Big[J_{012}(2,2,1)+J_{012}(1,2,2)\Big]$$
$$+\frac{(3n-11)M^2}{(3n-10)(n-3)}J_c+A_0(m)A_0(M)\frac{(n-2)^2(2n-7)}{4M^2m^2(3n-10)(n-3)}\ , \tag{3.20}$$

where $A_0(m)$ is defined in (3.10).

3.2 V_{0012}

Let us consider the second diagram of Fig. 3,

$$V_{00mM}(\lambda,\alpha,\beta,\sigma)=\int\frac{d^nk_1d^nk_2}{((p-k_1)^2)^\lambda((k_1-k_2)^2)^\alpha(k_2^2+m^2)^\beta(k_1^2+M^2)^\sigma}\ , \tag{3.21}$$

with the external momentum on the mass–shell, $p^2=-m^2$. For this class of integrals we have only one master-integral, the one with indices $(1,1,1,1)$. For constructing the ε-expansion we may use the differential equation method [32]. In terms of the new variable $r=m^2/M^2$ the result may be represented as

$$V_{00mM}=(m^2)^{-2\varepsilon}\left(\frac{1}{2\varepsilon^2}+\frac{S(r)}{\varepsilon}+F(r)+\varepsilon E(r)+\mathcal{O}(\varepsilon^2)\right)\ . \tag{3.22}$$

The analytical results for arbitrary r read

$$\begin{aligned}
S(r) &= \frac{5}{2}+\frac{(1-r)\ln(1-r)}{r}+\ln r\ ,\\
F(r) &= \frac{19}{2}-2\frac{1-r}{r}\mathrm{Li}_2(r)+\zeta_2(1-3r)+4\ln r+\left(1-\frac{r}{2}\right)\ln^2 r\\
&\quad +\frac{(1-r)^2}{r}\ln r\ln(1-r)+\frac{4(1-r)}{r}\ln(1-r)-\frac{(1-r)(3-r)}{2r}\ln^2(1-r)\ ,\\
E(r) &= \frac{65}{2}-\frac{(1-r)(3r-7)}{r}\mathrm{S}_{1,2}(r)+\frac{(1-r)(3r-5)}{r}\mathrm{Li}_3(r)\\
&\quad +\frac{2(1-r)(3-r)}{r}\mathrm{Li}_2(r)\ln(1-r)-\frac{1-r^2}{r}\mathrm{Li}_2(r)\ln r\\
&\quad -\frac{2(1-r)^2}{r}\ln r\ln^2(1-r)+\frac{(1-r)(1-2r)}{r}\ln^2 r\ln(1-r)\\
&\quad +\frac{(4-3r)}{6}\ln^3 r+\frac{(1-r)(5-3r)}{3r}\ln^3(1-r)-(1+3r)\zeta_3\\
&\quad +2(1-r)\zeta_2\ln r-\frac{(1-r)(5r-3)}{r}\zeta_2\ln(1-r)-\frac{8(1-r)}{r}\mathrm{Li}_2(r)\\
&\quad +\frac{2(1-r)(r-3)}{r}\ln^2(1-r)+\frac{4(1-r)^2}{r}\ln r\ln(1-r)+2(2-r)\ln^2 r\\
&\quad +6(1-2r)\zeta_2+\frac{12(1-r)}{r}\ln(1-r)+12\ln r\ .
\end{aligned} \tag{3.23}$$

For $r = 1$ (on–shell case) we have:

$$S(1) = \frac{5}{2}\,, \quad F(1) = \frac{19}{2} - 2\zeta_2\,, \quad E(1) = \frac{65}{2} - 6\zeta_2 - 4\zeta_3\,.$$

3.3 V_{1112}

The last and most complicated master–integral is

$$V_{mmmM}(\alpha,\beta,\sigma,\lambda) = \int \frac{d^nk_1 d^nk_2}{((p-k_1)^2+m^2)^\sigma((k_1-k_2)^2+m^2)^\alpha(k_2^2+m^2)^\beta(k_1^2+M^2)^\lambda} \tag{3.24}$$

where the external momentum is on the mass–shell, $p^2 = -m^2$. Let us introduce here an angle θ defined via [33]

$$\cos\theta = \frac{M}{2m}\,, M \le 2m\,. \tag{3.25}$$

The ε-expansion now can be written as

$$V_{mmmM}(\theta) = (m^2)^{-2\varepsilon}\left[\frac{1}{2\varepsilon^2} + \frac{1}{\varepsilon}S(\theta) + F(\theta) + \varepsilon E(\theta) + \mathcal{O}(\varepsilon^2)\right]. \tag{3.26}$$

The analytical results for arbitrary values of the angle θ read

$$\begin{aligned}
S(\theta) &= \frac{5}{2} - 4\theta\cos\theta\sin\theta - 4\cos^2\theta\ln(2\cos\theta)\,,\\
F(\theta) &= \frac{19}{2} + 4\theta\sin(2\theta)\ln(2\sin 2\theta) - 8\theta\sin(2\theta) - 4\theta^2\sin^2\theta + 2\zeta_2\sin^2\theta\\
&\quad -\sin(2\theta)\Big[4\mathrm{Ls}_2(2\theta) - 3\mathrm{Ls}_2(4\theta)\Big] + 4\cos^2\theta\ln^2(2\cos\theta) - 16\cos^2\theta\ln(2\cos\theta)\,,\\
E(\theta) &= \frac{65}{2} - 24\theta\sin(2\theta) - 16\theta^2\sin^2\theta + 12\mathrm{Cl}_3(2\theta)\sin^2\theta - 3\mathrm{Cl}_3(4\theta)\sin^2\theta\\
&\quad +16\theta\sin(2\theta)\ln(2\sin(2\theta)) + 16\theta^2\sin^2\theta\ln(2\sin\theta) - 4\theta\sin(2\theta)\ln^2(2\sin(2\theta))\\
&\quad +8\sin(2\theta)\ln(2\sin(2\theta))\mathrm{Ls}_2(2\theta) - 6\sin(2\theta)\ln(2\sin(2\theta))\mathrm{Ls}_2(4\theta)\\
&\quad -48\ln(2\cos\theta) + 8\theta^2\sin^2\theta\ln(2\cos\theta) + 48\sin^2\theta\ln(2\cos\theta)\\
&\quad +16\cos^2\theta\ln^2(2\cos\theta) - \frac{8}{3}\cos^2\theta\ln^3(2\cos\theta) + 8\theta\sin^2\theta\mathrm{Ls}_2(2\theta)\\
&\quad -16\sin(2\theta)\mathrm{Ls}_2(2\theta) + 12\sin(2\theta)\mathrm{Ls}_2(4\theta) - 4\sin(2\theta)\mathrm{Ls}_3(2\theta) + 3\sin(2\theta)\mathrm{Ls}_3(4\theta)\\
&\quad -12\zeta_2\sin^2\theta\ln(2\cos\theta) - 4\zeta_2 + 8\zeta_2\sin^2\theta - 2\zeta_3\sin^2\theta\,,
\end{aligned} \tag{3.27}$$

where the $\mathrm{Ls}_j(\theta)$ are so-called log-sine integrals [35] defined by

$$\mathrm{Ls}_j(\theta) = -\int\limits_0^\theta \mathrm{d}\phi\ \ln^{j-1}\left|2\sin\frac{\phi}{2}\right|\,, \tag{3.28}$$

$\mathrm{Cl}_j(\theta)$ are Clausen functions,

$$\mathrm{Cl}_j(\theta) = \begin{cases} \frac{1}{2\mathrm{i}}\left[\mathrm{Li}_j\left(e^{\mathrm{i}\theta}\right) - \mathrm{Li}_j\left(e^{-\mathrm{i}\theta}\right)\right], & j \text{ even} \\ \frac{1}{2}\left[\mathrm{Li}_j\left(e^{\mathrm{i}\theta}\right) + \mathrm{Li}_j\left(e^{-\mathrm{i}\theta}\right)\right], & j \text{ odd} \end{cases} \tag{3.29}$$

and the Li_j are poly-logarithms.

For $M^2 = m^2$ the angle θ is equal to $\pi/3$ and the integral is equal to the single scale diagram **V1111** calculated in [36]. In this case we obtain

$$\begin{aligned} S\left(\frac{\pi}{3}\right) &= \frac{5}{2} - \frac{\pi}{\sqrt{3}}, \quad F\left(\frac{\pi}{3}\right) = \frac{19}{2} + \frac{\pi}{\sqrt{3}}\ln 3 - 4\frac{\pi}{\sqrt{3}} - \frac{1}{2}\zeta_2 - 7\frac{\mathrm{Ls}_2\left(\frac{\pi}{3}\right)}{\sqrt{3}}, \\ E\left(\frac{\pi}{3}\right) &= \frac{65}{2} - 6\zeta_2 + 4\frac{\pi}{\sqrt{3}}\ln 3 - \frac{1}{2}\frac{\pi}{\sqrt{3}}\ln^2 3 - 12\frac{\pi}{\sqrt{3}} - \frac{9}{2}\frac{\pi}{\sqrt{3}}\zeta_2 + 4\zeta_2\ln 3 \\ &\quad -\frac{9}{2}\zeta_3 + 7\frac{\mathrm{Ls}_2\left(\frac{\pi}{3}\right)}{\sqrt{3}}\ln 3 - 28\frac{\mathrm{Ls}_2\left(\frac{\pi}{3}\right)}{\sqrt{3}} + \frac{4}{3}\pi\mathrm{Ls}_2\left(\frac{\pi}{3}\right) - \frac{21}{2}\frac{\mathrm{Ls}_3\left(\frac{2\pi}{3}\right)}{\sqrt{3}}. \end{aligned} \tag{3.30}$$

Another case of interest is the massless case $M^2 = 0$, which corresponds to $\theta = \pi/2$. In this case the diagram can be reduced to $J_{mmm}(1,1,1)$ [37, 38] plus a combination of simpler vacuum diagrams [39]. The results here are

$$S\left(\frac{\pi}{2}\right) = \frac{5}{2}, \quad F\left(\frac{\pi}{2}\right) = \frac{19}{2} - 4\zeta_2, \quad E\left(\frac{\pi}{2}\right) = \frac{65}{2} + 24\zeta_2\ln 2 - 20\zeta_2 - 14\zeta_3.$$

The expression (3.27) is directly applicable in the region $M \le 2m$ only. For $(M > 2m)$ one has to perform the proper analytical continuation. How this can be done is described in details in [41, 34, 31]. For this purpose, let us introduce the new variable

$$y \equiv e^{\mathrm{i}\sigma 2\theta}, \qquad \ln(-y - \mathrm{i}\sigma 0) = \ln y - \mathrm{i}\sigma\pi. \tag{3.31}$$

which coincides with the variable y defined in (3.19). In terms of this variable the expression (3.27) may be written as

$$\begin{aligned} S(y) &= \frac{5}{2} + (1+y)\ln y - \frac{(1+y)^2}{y}\ln(1+y), \\ F(y) &= \frac{19}{2} + \frac{1-y^2}{y}\Big[\mathrm{Li}_2(y) + 3\mathrm{Li}_2(-y)\Big] + \ln^2 y + (1-y)\zeta_2 \\ &\quad +(1+y)\Big(4\ln y - 2\ln y\ln(1-y) - 2\ln y\ln(1+y)\Big) \\ &\quad +\frac{(1+y)^2}{y}\Big[\ln^2(1+y) - 4\ln(1+y) + \ln y\ln(1-y)\Big], \\ E(y) &= \frac{65}{2} + \frac{1-y^2}{y}\Big[4\mathrm{S}_{1,2}(y) - 3\mathrm{S}_{1,2}\left(y^2\right) - 6\ln(1-y)\mathrm{Li}_2(-y) - 2\ln(1-y)\mathrm{Li}_2(y) \end{aligned}$$

$$
\begin{aligned}
&+4\mathrm{Li}_3\,(y) - 6\ln(1+y)\mathrm{Li}_2\,(-y) - 2\ln(1+y)\mathrm{Li}_2\,(y) - 2\ln y\ln(1-y)\ln(1+y)\\
&\left.-\ln y\ln^2(1-y) - \zeta_2\ln(1-y) + 12\mathrm{Li}_2\,(-y) + 4\mathrm{Li}_2\,(y) + 4\ln y\ln(1-y)\right]\\
&-12\zeta_2\ln(1+y) + 4\ln^2 y + 2\frac{(1+2y)(1+y)}{y}\zeta_2\ln(1+y) - 4\ln^2 y\ln(1-y)\\
&-2\ln^2 y\ln(1+y) + \ln^3 y + (1-y)(2\zeta_2\ln y - \zeta_3) + 4y\ln y\mathrm{Li}_2\,(y) - 4y\zeta_2\\
&+(1+y)\left[12\ln y - 8\ln y\ln(1+y) + 2\ln y\ln^2(1+y) - \frac{1}{3}\ln^3 y\right]\\
&+\frac{(1+y)^2}{y}\left[4\ln^2(1+y) - 12\ln(1+y) - \frac{2}{3}\ln^3(1+y) + \ln^2 y\ln(1-y)\right.\\
&\left.-\ln y\mathrm{Li}_2\,(y)\right] - \frac{3}{y}(y-1)(y+3)\mathrm{Li}_3\,(-y)\ . \qquad (3.32)
\end{aligned}
$$

The results (3.16)-(3.20),(3.23) (3.32) have been checked by a heavy mass expansion [24] with the help of the packages described in [40].

4 Renormalization

The mass renormalization constant Z_t in the $\overline{\mathrm{MS}}$ scheme at two loops may be written in the form

$$
\begin{aligned}
m_{t,0} &= m_t(\mu^2)\,Z_t = m_t(\mu^2)\left(1 + \frac{g^2(\mu^2)}{16\pi^2}\,\frac{1}{\varepsilon}Z_\alpha^{(1,1)} + \frac{\alpha_s(\mu^2)}{4\pi}\,\frac{1}{\varepsilon}Z_{\alpha_s}^{(1,1)}\right.\\
&\left.+\frac{\alpha_s(\mu^2)}{4\pi}\frac{g^2(\mu^2)}{16\pi^2}\left(\frac{1}{\varepsilon}Z_{\alpha\alpha_s}^{(2,1)} + \frac{1}{\varepsilon^2}Z_{\alpha\alpha_s}^{(2,2)}\right) + \mathcal{O}(g^4,\alpha_s^2)\right), \qquad (4.33)
\end{aligned}
$$

where $\alpha_s = g_s^2/4\pi$. Note that in contrast to the corresponding definition for bosons (used, e.g., in [18, 19]), the mass renormalization constant here (i.e., for fermions) relates the masses and not the squared ones. The coefficient $Z_\alpha^{(1,1)}$ may be extracted from Eq.(4.31) of [19] ($2Z_\alpha^{(1,1)} = Z_{\mathrm{t-quark}}^{(1,1)}$). The coefficient $Z_{\alpha_s}^{(1,1)}$ is well known [10]. For completeness we give them here:

$$
\begin{aligned}
Z_\alpha^{(1,1)} &= \frac{1}{3} - \frac{1}{3}\frac{m_Z^2}{m_W^2} - \frac{3}{4}\frac{m_Z^4}{m_W^2 m_H^2} - \frac{3}{8}\frac{m_H^2}{m_W^2} - \frac{3}{2}\frac{m_W^2}{m_H^2} + \frac{3}{8}\frac{m_t^2}{m_W^2} + N_c\frac{m_t^4}{m_W^2 m_H^2}\ ,\\
Z_{\alpha_s}^{(1,1)} &= -3C_f. \qquad (4.34)
\end{aligned}
$$

In our calculation we obtained the two-loop renormalization constants $Z_{\alpha\alpha_s}^{(2,1)}$ and $Z_{\alpha\alpha_s}^{(2,2)}$

$$
Z_{\alpha\alpha_s}^{(2,2)} = C_f\left[\frac{m_Z^2}{m_W^2} + \frac{9}{4}\frac{m_Z^4}{m_W^2 m_H^2} - 9N_c\frac{m_t^4}{m_H^2 m_W^2} + \frac{9}{8}\frac{m_H^2}{m_W^2} - \frac{9}{4}\frac{m_t^2}{m_W^2} + \frac{9}{2}\frac{m_W^2}{m_H^2} - 1\right], \qquad (4.35)
$$

$$
Z_{\alpha\alpha_s}^{(2,1)} = C_f\left[2N_c\frac{m_t^4}{m_W^2 m_H^2} + \frac{3}{2}\frac{m_t^2}{m_W^2} + \frac{19}{48}\frac{m_Z^2}{m_W^2} + \frac{31}{24}\right], \qquad (4.36)
$$

where, in the SM, $C_f = 4/3$, $N_c = 3$ and the first five quarks are treated as massless. The terms proportional to m_t^4 are coming from the tadpole contribution and will cancel in observable quantities. We may use the SM renormalization group equations to cross-check the $1/\varepsilon^2$– and $1/\varepsilon$–terms (see also [18, 19]). The coefficient (4.35) of the $1/\varepsilon^2$ term may be calculated from the relations

$$\left(\gamma_t + \sum_j \beta_{g_j} \frac{\partial}{\partial g_j} + \sum_i \left[\mu^2 \frac{\partial}{\partial \mu^2} m_i^2(\mu^2)\right] \frac{\partial}{\partial m_i^2}\right) Z_t^{(1)} \quad = \quad \frac{1}{2} \sum_j g_j \frac{\partial}{\partial g_j} Z_t^{(2)}, \qquad (4.37)$$

where we adopted the notation $m_{t,0} = m_t \left(1 + \sum_k Z_t^{(k)}/\varepsilon^k\right)$ and $g_j = g, g_s$. The anomalous dimension of the top-quark mass γ_t is defined by

$$\gamma_t \equiv \frac{1}{m_t} \mu^2 \frac{\partial}{\partial \mu^2} m_t(\mu^2) = \frac{1}{2} \sum_j g_j \frac{\partial}{\partial g_j} Z_t^{(1)} \ . \qquad (4.38)$$

Translated into a relation for the coefficients $Z^{(i,j)}$ defined in (4.33) we have

$$\begin{aligned}
\gamma_t &= \frac{1}{16\pi^2}\left(g^2 Z_\alpha^{(1,1)} + g_s^2 Z_{\alpha_s}^{(1,1)}\right) + \frac{g^2 g_s^2}{(16\pi^2)^2} 2 Z_{\alpha\alpha_s}^{(2,1)} + \mathcal{O}(g^4, \alpha_s^2) \ , \\
Z_{\alpha\alpha_s}^{(2,2)} &= Z_{\alpha_s}^{(1,1)} \left(1 + m_t^2 \frac{\partial}{\partial m_t^2}\right) Z_\alpha^{(1,1)} = Z_{\alpha_s}^{(1,1)} \left(Z_\alpha^{(1,1)} + \frac{3}{8}\frac{m_t^2}{m_W^2} + 2N_c \frac{m_t^4}{m_W^2 m_H^2}\right) . \qquad (4.39)
\end{aligned}$$

Note that (4.39) explicitly reveals, that the systematic inclusion of the tadpole contributions is important for the self–consistency of the RG equations.

The terms proportional to $1/\varepsilon$ may be deduced from the RG equations calculated in the unbroken phase. It has been shown in [18, 19] (details are given in [42]) that in the $\overline{\mathrm{MS}}$ scheme we may write

$$m_t^2(\mu^2) = \frac{1}{2} \frac{Y_t^2(\mu^2)}{\lambda(\mu^2)} \ m^2(\mu^2) \ , \qquad (4.40)$$

where m^2 and λ are the parameters of the symmetric scalar potential and Y_t is the top-quark Yukawa coupling. As a consequence we get the following relation for the anomalous dimension of the mass of the top-quark γ_t

$$\gamma_t = \gamma_Y + \frac{1}{2}\gamma_{m^2} - \frac{1}{2}\frac{\beta_\lambda}{\lambda} \ , \qquad (4.41)$$

where the relevant RG results may be found in [43]:

$$\begin{aligned}
\gamma_{m^2} &\equiv \frac{1}{m^2}\mu^2 \frac{\partial}{\partial \mu^2} m^2 = \frac{1}{16\pi^2}\left[\lambda + 3Y_t^2 - \frac{9}{4}g^2 - \frac{3}{4}g'^2\right] + 20\frac{Y_t^2 g_s^2}{(16\pi^2)^2} + \mathcal{O}(g^4) \ , \\
\gamma_Y &\equiv \frac{1}{Y_t}\mu^2 \frac{\partial}{\partial \mu^2} Y_t = \frac{1}{16\pi^2}\left[\frac{9}{4}Y_t^2 - 4g_s^2 - \frac{9}{8}g^2 - \frac{17}{24}g'^2\right] \\
&\qquad + \frac{g_s^2}{(16\pi^2)^2}\left[18Y_t^2 + \frac{9}{2}g^2 + \frac{19}{18}g'^2\right] + \mathcal{O}(g^4) \ ,
\end{aligned}$$

$$\beta_\lambda \equiv \mu^2 \frac{\partial}{\partial \mu^2}\lambda = \frac{1}{(16\pi^2)}\left[2\lambda^2 + 6\lambda Y_t^2 - 18 Y_t^4 - \frac{9}{2}\lambda g^2 - \frac{3}{2}\lambda g'^2 + \frac{27}{8}g^4 + \frac{9}{4}g^2 g'^2 + \frac{9}{8}g'^4\right]$$
$$+\frac{g_s^2 Y_t^2}{(16\pi^2)^2}\left[40\lambda - 96 Y_t^2\right] + \mathcal{O}(g^4) \ . \tag{4.42}$$

Finally, the parameter relations

$$Y_t^2 = \frac{2m_t^2}{v^2}\ , \quad \lambda = \frac{3m_H^2}{v^2}\ , \quad g^2 = \frac{4m_W^2}{v^2}\ , \quad g'^2 = \frac{4(m_Z^2 - m_W^2)}{v^2}\ ,$$

provide the necessary bridge between Eqs. (4.41) and (4.42) and our Eqs. (4.34) and (4.36).

We now turn to the discussion of the pole–mass relation (2.9). The calculation of the one-loop $\overline{\mathrm{MS}}$ renormalized on–shell amplitude Σ_1 (see (2.7) for the definition) is simple. We get it by rewriting the bare expression in terms of $\overline{\mathrm{MS}}$ parameters. In terms of the amplitudes X defined by

$$\Sigma_{1,0} = \frac{g_{s,0}^2}{16\pi^2} X^{(1)}_{\alpha_s,0} + \frac{g_0^2}{16\pi^2} X^{(1)}_{\alpha,0} \tag{4.43}$$

we obtain

$$\begin{aligned}
\left\{\Sigma_1\right\}_{\overline{\mathrm{MS}}} &\equiv \lim_{\varepsilon\to 0}\left(\frac{m_{0,t}}{m_t(\mu)}\left[1 + \frac{g_{s,0}^2}{16\pi^2}X^{(1)}_{\alpha_s,0} + \frac{g_0^2}{16\pi^2}X^{(1)}_{\alpha,0}\right] - 1\right) \\
&= \lim_{\varepsilon\to 0}\left(\frac{\alpha_s}{4\pi}\left[X^{(1)}_{\alpha_s,0} + \frac{1}{\varepsilon}Z^{(1,1)}_{\alpha_s}\right] + \frac{e^2}{16\pi^2\sin^2\theta_W}\left[X^{(1)}_{\alpha,0} + \frac{1}{\varepsilon}Z^{(1,1)}_{\alpha}\right]\right) \\
&= \frac{\alpha_s}{4\pi}\, X^{(1)}_{\alpha_s} + \frac{e^2}{16\pi^2\sin^2\theta_W}\, X^{(1)}_{\alpha}\ ,
\end{aligned} \tag{4.44}$$

where $X^{(1)}_{\alpha_s} = 4C_f + Z^{(1,1)}_{\alpha_s}\ln\frac{m_t^2}{\mu^2}$ and $X^{(1)}_{\alpha} = \frac{1}{2}X^{(1)}_{top}$ where $X^{(1)}_{top}$ is given in Eq. (B.5) of Ref. [19]. We note that the explicit μ–dependence is given by the following structure:

$$X^{(1)}_i = \Delta X^{(1)}_i + Z^{(1,1)}_i \ln\frac{m_t^2}{\mu^2}\ \ ; \ \ (i = \alpha, \alpha_s) \tag{4.45}$$

where $\Delta X^{(1)}_i$ does not explicitly depend on μ. At the two-loop level, we may avoid the consideration of the wave-function renormalization as well as the renormalization of the ghost sector and of the gauge parameters if we look directly at the full two-loop $\overline{\mathrm{MS}}$ renormalized on–shell amplitude. The latter can be written in the form

$$\begin{aligned}
&\left\{\Sigma_2 + \Sigma_1\Sigma_1'\right\}_{\overline{\mathrm{MS}}} = \lim_{\varepsilon\to 0}\Bigg(\Sigma_{2,0} + \Sigma_{1,0}\Sigma_{1,0}' + \frac{\alpha_s}{4\pi}\frac{e^2}{16\pi^2\sin^2\theta_W}\left[\frac{1}{\varepsilon}Z^{(2,1)}_{\alpha\alpha_s} + \frac{1}{\varepsilon^2}Z^{(2,2)}_{\alpha\alpha_s}\right] \\
&+\frac{\alpha_s}{4\pi}\frac{e^2}{16\pi^2\sin^2\theta_W}\ \frac{1}{\varepsilon}\left\{Z^{(1,1)}_{\alpha}\left[1 + 2m_t^2\frac{\partial}{\partial m_{t,0}^2}\right]X^{(1)}_{\alpha_s,0} + Z^{(1,1)}_{\alpha_s}\left[1 + 2m_t^2\frac{\partial}{\partial m_{t,0}^2}\right]X^{(1)}_{\alpha,0}\right\}\Bigg) \\
&= \frac{\alpha_s}{4\pi}\frac{e^2}{16\pi^2\sin^2\theta_W}\Bigg(C^{(2,2)}_{\alpha\alpha_s}\ln^2\frac{m_t^2}{\mu^2} + C^{(2,1)}_{\alpha\alpha_s}\ln\frac{m_t^2}{\mu^2}
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{8}\ln\left(1-\frac{1}{\omega_t}\right)(1-\omega_t)\,\frac{(18\omega_t^2+21\omega_t+17)}{\omega_t}+\ln\omega_t\left[\frac{22\omega_t+17}{8\omega_t}\right]\\
&+(1-\omega_t)\,\frac{(1+2\omega_t)(2+\omega_t)}{2\omega_t}\ln\omega_t\ln\left(1-\frac{1}{\omega_t}\right)+\frac{1+\omega_t-\omega_t^2}{2\omega_t}\ln^2\omega_t\\
&-(1-\omega_t)^2\frac{4\omega_t+5}{8\omega_t}\ln^2\left(1-\frac{1}{\omega_t}\right)+\frac{(1+\omega_t)}{4\omega_t}\left(4\omega_t^2+7\omega_t-9\right)\mathrm{Li}_2\left(\frac{1}{\omega_t}\right)\\
&-\frac{1}{\omega_t}(1-\omega_t)^2(1+2\omega_t)\left\{\frac{3}{2}\mathrm{S}_{1,2}\left(\frac{1}{\omega_t}\right)-\frac{3}{2}\mathrm{Li}_3\left(\frac{1}{\omega_t}\right)-\ln\omega_t\mathrm{Li}_2\left(\frac{1}{\omega_t}\right)\right.\\
&\left.+\frac{1}{2}\ln\left(1-\frac{1}{\omega_t}\right)\mathrm{Li}_2\left(\frac{1}{\omega_t}\right)+\frac{1}{4}\ln\left(1-\frac{1}{\omega_t}\right)\left[\ln^2\omega_t+6\zeta_2\right]\right\}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{(1+y_Z)^2(1+y_Z^2)(17+41y_Z+17y_Z^2)}{18\omega_t y_Z^3}\left\{3\left(2\mathrm{Li}_3\left(y_Z\right)+\mathrm{Li}_3\left(-y_Z\right)\right)-3\zeta_2\ln(1+y_Z)\right.\\
&\qquad\left.-2\ln y_Z\left(2\mathrm{Li}_2\left(y_Z\right)+\mathrm{Li}_2\left(-y_Z\right)\right)-\ln^2 y_Z\left(\ln(1-y_Z)+\frac{1}{2}\ln(1+y_Z)\right)\right\}\\
&+\frac{(1-y_Z)(1+y_Z)^3(17+41y_Z+17y_Z^2)}{18\omega_t y_Z^3}\times\\
&\qquad\left\{2\ln y_Z\left(\ln(1-y_Z)+\frac{1}{2}\ln(1+y_Z)\right)+2\mathrm{Li}_2\left(y_Z\right)+\mathrm{Li}_2\left(-y_Z\right)\right\}
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{4}{9}\frac{(1+y_Z)}{y_Z^2}\left(5-4\frac{\omega_t y_Z}{(1+y_Z)^2}\right)\times\\
&\left\{\frac{(1+y_Z^2)(1+4y_Z+y_Z^2)}{(1+y_Z)}\left[3\left(2\mathrm{Li}_3\left(y_Z\right)+\mathrm{Li}_3\left(-y_Z\right)\right)-3\zeta_2\ln(1+y_Z)\right.\right.\\
&\qquad\left.-2\ln y_Z\left(2\mathrm{Li}_2\left(y_Z\right)+\mathrm{Li}_2\left(-y_Z\right)\right)-\ln^2 y_Z\left(\ln(1-y_Z)+\frac{1}{2}\ln(1+y_Z)\right)\right]\\
&+(1-y_Z)(1+4y_Z+y_Z^2)\left[2\ln y_Z\left(\ln(1-y_Z)+\frac{1}{2}\ln(1+y_Z)\right)+2\mathrm{Li}_2\left(y_Z\right)\right]\\
&+\frac{9}{4}(1-y_Z)^2(1+y_Z)\ln(1+y_Z)+\frac{(1+2y_Z-24y_Z^2+2y_Z^3+y_Z^4)}{(1-y_Z)}\mathrm{Li}_2\left(-y_Z\right)\\
&\left.+\frac{3}{4}\frac{y_Z(1+3y_Z)(4-y_Z+y_Z^2)}{(1-y_Z)}\ln y_Z-\frac{1}{4}\frac{y_Z(2+9y_Z+3y_Z^2+16y_Z^3+6y_Z^4)}{(1+y_Z)(1-y_Z)}\ln^2 y_Z\right\}
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{(1+y_Z)^3(9+32y_Z+9y_Z^2)}{4\omega_t(1-y_Z)y_Z^2}\mathrm{Li}_2\left(-y_Z\right)+\frac{447}{16}+\frac{125}{9}\frac{(1+y_Z^2)}{y_Z}\\
&+\frac{32}{3}\left[1-\omega_t\frac{y_Z}{(1+y_Z)^2}\right]\left\{\zeta_3-4\zeta_2\ln 2\right\}+\frac{(1+y_Z)^4(17y_Z^2-19y_Z+17)}{8\omega_t y_Z^3}\ln(1+y_Z)
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{1}{\omega_t}\ln^2 y_Z\left\{-\frac{685}{36}+\frac{17}{36y_Z^2}+\frac{67}{24y_Z}-\frac{335}{24}y_Z-\frac{497}{72}y_Z^2-\frac{17}{12}y_Z^3+\frac{25}{1-y_Z}\right\}\\
&+\frac{(1+y_Z)}{24\omega_t y_Z^2(1-y_Z)}\left[51y_Z^5+113y_Z^4+134y_Z^3+237y_Z^2+197y_Z+68\right]\ln y_Z\\
&-\frac{1}{3}\zeta_2\left\{\frac{20-39y_Z-y_Z^2-80y_Z^3-20y_Z^4}{y_Z(1-y_Z)}-\omega_t\frac{7-49y_Z+17y_Z^2-55y_Z^3-16y_Z^4)}{(1-y_Z)(1+y_Z)^2}\right\}\\
&-\frac{1}{\omega_t}\left\{\frac{4157}{72}+\frac{425(1+y_Z^4)}{72y_Z^2}+\frac{2561(1+y_Z^2)}{96y_Z}\right\}\\
&+\frac{1}{\omega_t}\zeta_2\left\{\frac{187}{3}+\frac{17}{6y_Z^2}+\frac{133}{12y_Z}+\frac{535}{12}y_Z+\frac{211}{12}y_Z^2+\frac{17}{6}y_Z^3-\frac{50}{1-y_Z}\right\}\\
&\\
&-3\omega_t\ln\omega_t\frac{(1+y_H+y_H^2)}{(1+y_H)^2}+\frac{3}{2\omega_t}\frac{y_H}{(1+y_H)^2}\frac{(1+y_Z)^4}{y_Z^2}\ln\frac{(1+y_Z)^2}{y_Z}\\
&-\frac{1}{\omega_t}\frac{y_H}{(1+y_H)^2}\left\{\frac{11(1+y_H^2)(1+y_H)^2}{8y_H^2}+8N_c+\frac{1}{2}\frac{(1+y_Z)^4}{y_Z^2}\right\}\\
&+\frac{1}{\omega_t}\zeta_2\left\{\frac{3}{2y_H}+\frac{9}{2}y_H+\frac{3}{4}y_H^2\right\}-\frac{\omega_t}{36}\frac{(625+1286y_H+625y_H^2)}{(1+y_H)^2}\\
&+\frac{1}{\omega_t}\frac{(1-y_H)^2}{y_H^2}\ln y_H\left[\ln(1-y_H)+\frac{1}{2}\ln(1+y_H)\right]\left[(1-y_H^2)-\frac{1}{2}(1+y_H^2)\ln y_H\right]\\
&-\frac{1}{8}\frac{1}{\omega_t}\frac{2+8y_H-10y_H^2-3y_H^3}{y_H}\ln^2 y_H+\frac{1}{8}\frac{1}{\omega_t}\frac{(1+y_H)(6-63y_H+5y_H^2)}{y_H}\ln y_H\\
&-\frac{1}{8}\frac{1}{\omega_t}\frac{(1+y_H)^2(5-62y_H+5y_H^2)}{y_H^2}\ln(1+y_H)-\frac{3}{2}\frac{1}{\omega_t}\zeta_2\ln(1+y_H)\frac{(1-y_H)^2(1+y_H^2)}{y_H^2}\\
&+\frac{1}{\omega_t}\frac{(1-y_H)(1+y_H)}{y_H^2}\left\{\frac{(5-28y_H+5y_H^2)}{4}\mathrm{Li}_2(-y_H)+(1-y_H)^2\mathrm{Li}_2(y_H)\right\}\\
&+\frac{1}{\omega_t}\frac{(1-y_H)^2(1+y_H^2)}{y_H^2}\left\{\frac{3}{2}\left[2\mathrm{Li}_3(y_H)+\mathrm{Li}_3(-y_H)\right]-\ln y_H\left[2\mathrm{Li}_2(y_H)+\mathrm{Li}_2(-y_H)\right]\right\}\Bigg)\,,
\end{aligned}
\tag{4.46}
$$

where

$$
\begin{aligned}
\omega_t &= \frac{m_W^2}{m_t^2}\,,\\
y_A &= \frac{1-\sqrt{1-\frac{4m_t^2}{m_A^2}}}{1+\sqrt{1-\frac{4m_t^2}{m_A^2}}}\,, A=H,Z\ .
\end{aligned}
\tag{4.47}
$$

We have explicitly factorized the RG logarithms, $C^{(2,1)}_{\alpha\alpha_s}$ and $C^{(2,2)}_{\alpha\alpha_s}$, which may be calculated also from the one-loop result and the knowledge of the mass anomalous dimensions (see [42]

for the general case):

$$
\begin{aligned}
C^{(2,2)}_{\alpha\alpha_s} &= Z^{(1,1)}_{\alpha_s}\left(1+m_t^2\frac{\partial}{\partial m_t^2}\right)Z^{(1,1)}_{\alpha} = Z^{(2,2)}_{\alpha\alpha_s} && (4.48)\\
C^{(2,1)}_{\alpha\alpha_s} &= 2Z^{(2,1)}_{\alpha\alpha_s} + 4Z^{(1,1)}_{\alpha}Z^{(1,1)}_{\alpha_s} + Z^{(1,1)}_{\alpha}\Delta X^{(1)}_{\alpha_s} + Z^{(1,1)}_{\alpha_s}\left(1+2m_t^2\frac{\partial}{\partial m_t^2}\right)\Delta X^{(1)}_{\alpha}\ , && (4.49)
\end{aligned}
$$

where

$$
\begin{aligned}
& m_t^2\frac{\partial}{\partial m_t^2}\Delta X^{(1)}_{\alpha} = \frac{(1+y_H)^4}{8\omega_t y_H^2}\ln(1+y_H) - \frac{(1+y_H)(3+y_H)}{8\omega_t}\ln y_H \\
& +\frac{1}{\omega_t}\left\{\frac{3}{4}\frac{(1+y_Z)^4}{y_Z^2}\frac{y_H}{(1+y_H)^2} + \frac{1}{4}\frac{(y_H^2+3y_H+1)}{y_H} - \frac{2y_H}{(1+y_H)^2}N_c\right\} \\
& +\omega_t\left[\frac{3}{2}\frac{y_H}{(1+y_H)^2} - \frac{25}{18}\right] + \frac{1}{\omega_t}\left\{\frac{(1+y_Z)^2}{3y_Z} - \frac{19}{4} - \frac{17(1+y_Z^4)}{36y_Z^2} - \frac{20(1+y_Z^2)}{9y_Z}\right\} \\
& -\frac{2}{9}\left(5-4\frac{\omega_t y_Z}{(1+y_Z)^2}\right)\left\{\frac{(1+y_Z)^4}{y_Z^2}\ln(1+y_Z) + \frac{y_Z(1+y_Z)(2+y_Z)}{(1-y_Z)}\ln y_Z\right\} \\
& +\frac{(1+y_Z)^3}{72\omega_t}\left[\frac{(1+y_Z)}{y_Z^3}(34+41y_Z+34y_Z^2)\ln(1+y_Z) + \frac{(41+34y_Z)}{(1-y_Z)}\ln y_Z\right] \\
& +\frac{169}{72} + \frac{10(1+y_Z^2)}{9y_Z} + \frac{1}{8}\ln\left(1-\frac{1}{\omega_t}\right)\frac{(1-\omega_t)(4\omega_t^2+\omega_t+1)}{\omega_t} + \frac{1}{8\omega_t}\ln\frac{1}{\omega_t}\ . \qquad (4.50)
\end{aligned}
$$

The $C^{(i,j)}$'s in the SM (in contrast to QCD) have non-polynomial structure in the dimensionless coupling constants which originates from the tadpole contributions.

5 Results

The relation between the top–propagator pole $\tilde{M}$ and the $\overline{\mathrm{MS}}$ mass m_t can be written as

$$
\frac{\tilde{M}}{m_t} = 1 + \left\{\Sigma_1\right\}_{\overline{\mathrm{MS}}} + \left\{\Sigma_2 + \Sigma_1\Sigma_1^{'}\right\}_{\overline{\mathrm{MS}}} + \mathcal{O}(g^4, \alpha_s^2)\ , \qquad (5.51)
$$

where the r.h.s. is given by Eqs. (4.44) and (4.46).

We would like to elucidate several aspects of our calculation: all diagrams have been calculated in the so-called modified $\overline{\mathrm{MS}}$ scheme ($\overline{\mathrm{MMS}}$) [37], which is defined by multiplying each loop by the factor $1/(4\pi)^\varepsilon/\Gamma(1+\varepsilon)$. It has been shown in [13, 18, 19] that at the two-loop level this scheme is equivalent to the $\overline{\mathrm{MS}}$ scheme when applied to calculating mass relations. We once more emphasize that the inclusion of the tadpole diagrams, shown in Fig. 2, is important for several reasons: i) to restore gauge invariance; ii) to get UV counterterms which coincide with the ones calculated in the unbroken phase; iii) to restore the proper form of the RG equations in the broken phase (the result (5.51) is manifestly RG invariant through $\mathcal{O}(g^4, \alpha_s^2)$).

As the top is an unstable particle the pole of the top–propagator exhibits a real part which is the pole–mass M_t and an imaginary part which up to a sign gives the width Γ_t divided by two (see (2.5) for the precise definitions). The imaginary part is coming from diagrams with $W, \phi^{\pm}$ lines (see Fig. 1) and may be calculated analytically. We find

$$
\begin{aligned}
\frac{\Gamma_t}{m_t} &= -2\,\mathrm{Im}\,\frac{\tilde{M}}{m_t} = \frac{e^2}{8\pi \sin^2\theta_W}\left\{\frac{1}{8\omega_t} - \frac{3}{8}\omega_t + \frac{1}{4}\omega_t^2\right\} \\
&+ \frac{\alpha_s}{4\pi}\frac{e^2}{8\pi\sin^2\theta_W}C_f\left\{\frac{9}{8}\ln\frac{m_t^2}{\mu^2}\left(2\omega_t^2 - \omega_t - \frac{1}{\omega_t}\right) + \frac{1}{8}(1-\omega_t)\omega_t\left(\frac{17}{\omega_t^2} + \frac{21}{\omega_t} + 18\right)\right. \\
&\quad -\frac{1}{4}\ln(1-\omega_t)(1-\omega_t)^2\left(\frac{5}{\omega_t} + 4\right) - \frac{1}{2}\ln\omega_t\left(1 - \omega_t - 2\omega_t^2\right) \\
&\quad \left. -\frac{1}{2}(1-\omega_t)^2\left(2 + \frac{1}{\omega_t}\right)\left[\ln\omega_t\ln(1-\omega_t) + 2\zeta_2 + 2\mathrm{Li}_2\left(\omega_t\right)\right]\right\}
\end{aligned}
\tag{5.52}
$$

as a result in terms of $\overline{\mathrm{MS}}$ parameters. With the help of (5.54) we may get the corresponding result in terms of on–shell parameters

$$
\begin{aligned}
\frac{\Gamma_t}{M_t} &= \frac{\alpha}{2\sin^2\theta_W^{OS}}\frac{1}{8}\left(1 - \frac{M_W^2}{M_t^2}\right)^2\left(1 + 2\frac{M_W^2}{M_t^2}\right)\frac{M_t^2}{M_W^2} \\
&- \frac{\alpha_s}{4\pi}\frac{\alpha}{2\sin^2\theta_W^{OS}}C_f\left\{\frac{1}{8}\left(1 - \frac{M_W^2}{M_t^2}\right)\frac{M_t^2}{M_W^2}\left(6\frac{M_W^4}{M_t^4} - 9\frac{M_W^2}{M_t^2} - 5\right)\right. \\
&\quad +\frac{1}{4}\ln\left(1 - \frac{M_W^2}{M_t^2}\right)\left(1 - \frac{M_W^2}{M_t^2}\right)^2\left(\frac{5M_t^2}{M_W^2} + 4\right) + \frac{1}{2}\ln\frac{M_W^2}{M_t^2}\left(1 - \frac{M_W^2}{M_t^2} - 2\frac{M_W^4}{M_t^4}\right) \\
&\quad \left. +\frac{1}{2}\left(1 - \frac{M_W^2}{M_t^2}\right)^2\left(2 + \frac{M_t^2}{M_W^2}\right)\left[\ln\frac{M_W^2}{M_t^2}\ln\left(1 - \frac{M_W^2}{M_t^2}\right) + 2\zeta_2 + 2\mathrm{Li}_2\left(\frac{M_W^2}{M_t^2}\right)\right]\right\},
\end{aligned}
\tag{5.53}
$$

which coincides with the well know result [44] for the tree and the $\mathcal{O}(\alpha_s)$ correction to the partial decay width for the process $t \to bW$ in the approximation of a massless b-quark.

Very often the inverse of the relation (5.51) is required. To that end we have to solve the real part of (2.9) iteratively for m_t and to express all $\overline{\mathrm{MS}}$ parameters in terms of on-shell ones. The solution to two loops reads

$$
\begin{aligned}
\frac{m_t}{M_t} &= 1 - \mathrm{Re}\left\{\Sigma_1\right\}_{\overline{\mathrm{MS}}} - \mathrm{Re}\left\{\Sigma_2 + \Sigma_1\Sigma_1'\right\}_{\overline{\mathrm{MS}}} + \left[\mathrm{Re}\left\{\Sigma_1\right\}_{\overline{\mathrm{MS}}}\right]^2 \\
&\quad \left. -\sum_j(\Delta m_j^2)^{(1)}\frac{\partial}{\partial m_j^2}\mathrm{Re}\left\{\Sigma_1\right\}_{\overline{\mathrm{MS}}} - (\Delta e)^{(1)}\frac{\partial}{\partial e}\mathrm{Re}\left\{\Sigma_1\right\}_{\overline{\mathrm{MS}}}\right|_{m_j^2=M_j^2,\, e=e_{\mathrm{OS}}},
\end{aligned}
\tag{5.54}
$$

where the sum runs over all species of particles $j = Z,\, W,\, H,\, t$ and

$$
(\Delta m_j^2)^{(1)} = -M_j^2\frac{e_{\mathrm{OS}}^2}{16\pi^2\sin^2\theta_W}X_j^{(1)}\Big|_{m_j^2=M_j^2}
$$

stands for the self-energy of the jth particle at $p^2=-m_j^2$ in the $\overline{\mathrm{MS}}$ scheme and parameters replaced by the on-shell ones. The values of $X_j^{(1)}$ are given in Appendix B of [18, 19]. The relation (5.54) includes also the transition from the $\overline{\mathrm{MS}}$ to on-shell scheme for the electric charge. It can be described as

$$(\Delta e)^{(1)}\frac{\partial}{\partial e}\mathrm{Re}\Big\{\Sigma_1\Big\}_{\overline{\mathrm{MS}}}=\Big\{\delta\alpha_{\mathrm{bos}}+\delta\alpha_{\mathrm{lep}}+\delta\alpha_{\mathrm{top}}+\Delta\alpha^{(5)}_{\mathrm{hadrons}}(M_Z^2)-\delta\Delta\alpha_{\mathrm{udscb}}(M_Z^2)\Big\}\,\mathrm{Re}\Big\{\Sigma_1\Big\}_{\overline{\mathrm{MS}}},$$

where

$$\delta\alpha_{\mathrm{bos}}=\frac{\alpha}{4\pi}\left(7\ln\frac{M_W^2}{\mu^2}-\frac{2}{3}\right),\quad \delta\alpha_{\mathrm{lep}}=-\frac{\alpha}{3\pi}\sum_{\ell=e,\mu,\tau}\ln\frac{m_\ell^2}{\mu^2},\quad \delta\alpha_{\mathrm{top}}=-\frac{4\alpha}{9\pi}\ln\frac{m_t^2}{\mu^2}$$

and

$$\delta\Delta\alpha_{\mathrm{udscb}}(M_Z^2)=\frac{11\alpha}{9\pi}\left(\ln\frac{M_Z^2}{\mu^2}-\frac{5}{3}\right).$$

For numerical estimations of $\Delta\alpha^{(5)}_{\mathrm{hadrons}}(M_Z^2)$ we use the results of [45]

$$\Delta\alpha^{(5)}_{\mathrm{hadrons}}(M_Z^2)=0.027773\pm0.000354\ ;\quad \alpha^{-1}(M_Z^2)=128.922\pm0.049 \tag{5.55}$$

at $M_Z=91.19$ GeV.

For the $\mathcal{O}(\alpha\alpha_s)$ contribution only the expression (5.54) actually simplifies to

$$\begin{aligned}\frac{m_t}{M_t} &= 1-\mathrm{Re}\Big\{\Sigma_1\Big\}_{\overline{\mathrm{MS}}}-\mathrm{Re}\Big\{\Sigma_2+\Sigma_1\Sigma_1'\Big\}_{\overline{\mathrm{MS}}}+2\frac{\alpha_s}{4\pi}\frac{e^2}{16\pi^2\sin^2\theta_W^{OS}}X_\alpha^{(1)}X_{\alpha_s}^{(1)}\\ &\quad+\frac{\alpha_s}{4\pi}\frac{e^2}{16\pi^2\sin^2\theta_W^{OS}}2\Big\{X_\alpha^{(1)}Z_{\alpha_s}^{(1,1)}+X_{\alpha_s}^{(1)}m_t^2\frac{\partial}{\partial m_t^2}X_\alpha^{(1)}\Big\},\end{aligned}\tag{5.56}$$

where

$$\begin{aligned}m_t^2\frac{\partial}{\partial m_t^2}X_\alpha^{(1)} &= m_t^2\frac{\partial}{\partial m_t^2}\Delta X_\alpha^{(1)}+\ln\frac{m_t^2}{\mu^2}\,m_t^2\frac{\partial Z_\alpha^{(1,1)}}{\partial m_t^2}+Z_\alpha^{(1,1)}\\ &= m_t^2\frac{\partial}{\partial m_t^2}\Delta X_\alpha^{(1,1)}+Z_\alpha^{(1,1)}+\ln\frac{m_t^2}{\mu^2}\left(\frac{3}{8}\frac{m_t^2}{m_W^2}+2N_c\frac{m_t^4}{m_W^2m_H^2}\right).\end{aligned}$$

For Eq.(5.56) we present a semi–numerical result for $\mu=M_t$ and the following input parameters: $\alpha=1/137.036$, $M_W=80.419$ GeV, $M_Z=91.188$ GeV and $M_t=174.3$ GeV,

$$\begin{aligned}\frac{m_t(M_t)}{M_t} &= 1-C_f\frac{\alpha_s}{\pi}-\frac{\alpha}{4\pi\sin^2\theta_W^{OS}}\Bigg(0.3747795+\frac{1}{2}\frac{M_W^2}{M_H^2}\left[1-3\ln\frac{M_W^2}{M_t^2}\right]\\ &\quad-\frac{3}{4}\frac{M_Z^4}{M_W^2M_H^2}\ln\frac{M_Z^2}{M_t^2}+\frac{M_t^4}{M_W^2M_H^2}\left\{\frac{1}{2}\frac{M_H^2}{M_t^2}\frac{(1+Y_H^2)}{Y_H}+\frac{1}{4}\frac{M_Z^4}{M_t^4}-3\right\}\\ &\quad-\frac{1}{8}\frac{M_H^4}{M_t^2M_W^2}\ln(1+Y_H)+\frac{1}{8}\frac{M_H^2}{M_W^2}\frac{(3+Y_H^2)}{(1+Y_H)}\ln Y_H\Bigg)\end{aligned}$$

$$
\begin{aligned}
&-C_f\frac{\alpha_s}{4\pi}\frac{\alpha}{4\pi\sin^2\theta_W^{OS}}\Biggl(-78.591+6\frac{M_W^2}{M_H^2}\ln\frac{M_W^2}{M_t^2}+3\frac{M_Z^4}{M_W^2M_H^2}\ln\frac{M_Z^2}{M_t^2}-2\frac{M_W^2}{M_H^2}\\
&+\frac{M_t^4}{M_H^2M_W^2}\left\{-\frac{11}{8}\frac{M_H^2}{M_t^2}\frac{(1+Y_H^2)}{Y_H}-\frac{M_Z^4}{M_t^4}+6\right\}+\zeta_2\frac{M_t^2}{M_W^2}\left\{\frac{3}{2Y_H}+\frac{9}{2}Y_H+\frac{3}{4}Y_H^2\right\}\\
&+\frac{M_t^2}{M_W^2}\frac{(1-Y_H)^2}{Y_H^2}\ln Y_H\left[\ln(1-Y_H)+\frac{1}{2}\ln(1+Y_H)\right]\left[(1-Y_H^2)-\frac{1}{2}(1+Y_H^2)\ln Y_H\right]\\
&-\frac{1}{8}\frac{M_t^2}{M_W^2}\frac{2+8Y_H-10Y_H^2-3Y_H^3}{Y_H}\ln^2 Y_H+\frac{1}{8}\frac{M_t^2}{M_W^2}(1+Y_H)(11Y_H-39)\ln Y_H\\
&-\frac{1}{8}\frac{M_H^2}{M_W^2}\frac{(11-50Y_H+11Y_H^2)}{Y_H}\ln(1+Y_H)-\frac{3}{2}\frac{M_t^2}{M_W^2}\zeta_2\ln(1+Y_H)\frac{(1-Y_H)^2(1+Y_H^2)}{Y_H^2}\\
&+\frac{M_t^2}{M_W^2}\frac{(1-Y_H)(1+Y_H)}{Y_H^2}\left\{\frac{(5-28Y_H+5Y_H^2)}{4}\mathrm{Li}_2\left(-Y_H\right)+(1-Y_H)^2\mathrm{Li}_2\left(Y_H\right)\right\}\\
&+\frac{M_t^2}{M_W^2}\frac{(1-Y_H)^2(1+Y_H^2)}{Y_H^2}\left\{\frac{3}{2}\Bigl[2\mathrm{Li}_3\left(Y_H\right)+\mathrm{Li}_3\left(-Y_H\right)\Bigr]-\ln Y_H\Bigl[2\mathrm{Li}_2\left(Y_H\right)+\mathrm{Li}_2\left(-Y_H\right)\Bigr]\right\}\Biggr),\quad(5.57)
\end{aligned}
$$

where Y_A is defined via the pole masses as

$$
Y_A=\frac{1-\sqrt{1-\frac{4M_t^2}{M_A^2}}}{1+\sqrt{1-\frac{4M_t^2}{M_A^2}}}\ ,A=H,Z\quad .
$$

For illustration of the numerical significance of our result we shown the two-loop $\mathcal{O}(\alpha\alpha_s)$ corrections to $M_t/m_t(m_t)$ and $m_t(M_t)/M_t$ (Fig. 4) as a function of the Higgs boson mass in comparison with the two- and three-loop QCD ones [15] (for simplicity, we consider the case $\alpha_s(\mu)=\alpha_s(M_Z)$). In a wide range of values of Higgs boson mass (100 GeV $< M_H <$ 1000 GeV) our result is comparable in size to the 3-loop QCD corrections. The corrections grow with the Higgs mass, again this is a consequence of the breakdown of the decoupling of heavy particles in a "spontaneously broken" gauge theory.

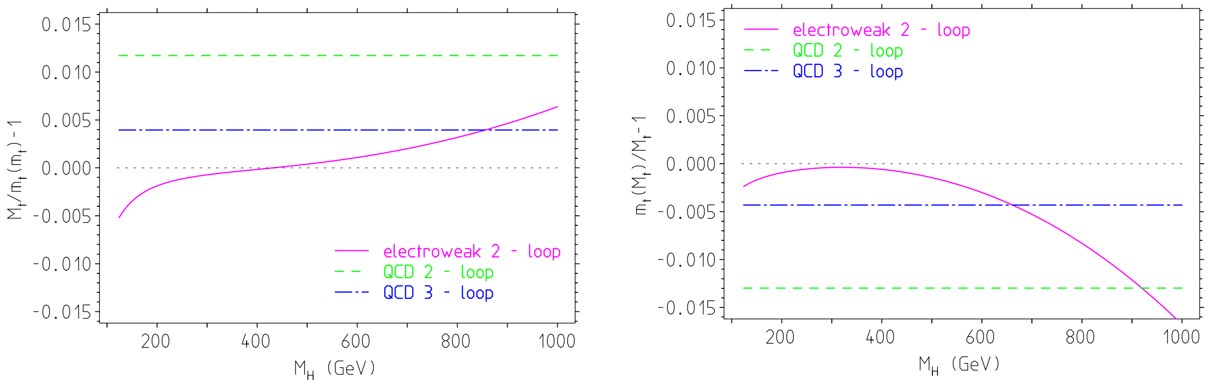

Figure 4: Electroweak $O(\alpha\alpha_s)$ correction to $M_t/m_t(m_t)-1$ [left] and $m_t(M_t)/M_t-1$ [right], in comparison with $O(\alpha_s^2)$ and $O(\alpha_s^3)$ QCD corrections as a function of the Higgs boson mass M_H.

6 Conclusion

The main result of the present investigation is the two-loop $\mathcal{O}(\alpha\alpha_s)$ relationship between pole– and $\overline{\text{MS}}$–mass for the top-quark within the SM. It is given by Eq. (4.46). Numerically its size is comparable to the 3-loop QCD–correction for a light Higgs boson and it increases for a heavy Higgs boson. As a byproduct, several new massive master-integrals have been calculated analytically (Sec. 3).

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